Plane Light-Like Shells and
Impulsive Gravitational Waves
in Scalar-Tensor Theories of Gravity

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Abstract

We study gravitational plane impulsive waves and electromagnetic shock waves in a scalar-tensor theory of gravity of the Brans-Dicke type. In vacuum, we present an exact solution of Brans-Dicke's field equations and give an example in which a plane impulsive gravitational wave and a null shell of matter coexist on the same hypersurface. In the homogenous case, we characterize them by their surface energy density and wave amplitude and discuss the inhomogenous case. We also give an exact solution of the Brans-Dicke's field equations in the electrovacuum case which admits a true curvature singularity and use it to built an example where a plane impulsive gravitational wave and an electromagnetic shock wave have the same null hypersurface as history of their wave fronts and propagate independently and decoupled from a null shell of matter. This last solution is shown to correspond to the space-time describing the interaction region resulting from the collision of two electromagnetic shock waves leading to the formation of two gravitational impulsive waves. The properties of this solution are discussed and compared to those of the Bell-Szekeres solution of general relativity.

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1 Introduction

Nearly all the attempts at deepening the connection between the gravitational interaction and the other interactions have led to the conclusion that scalar fields play an important role in the process of quantization. This idea first came with the scalar field (the compacton) arising from the compactification of the fifth dimension in Kaluza-Klein theory [1]. Later, the existence of a scalar field found its justification in the Brans-Dicke model [2] and its further generalizations including in particular non-linear σ-models with several scalar fields [3]. More recently, the low energy limit of string theory has been found to exhibit a scalar field coupled to gravity, the dilaton, having a local space-time dependent coupling to the matter fields [4] while the scalar fields in scalar-tensor theories of the Brans-Dicke type present a universal metric coupling to matter [5]. Scalar fields are also believed to be important in cosmology, for instance in the scenarios of inflation (inflaton) and extended inflation (inflaton and Brans-Dicke field) [6].

Plane wave geometries have many interesting properties in four dimensions such as being geodesically complete and admitting metrics belonging to the family of the pp-waves [7], [8] which are plane-fronted waves with parallel rays. They are known since a long time to be exact solutions of Einstein's vacuum field equations. More recently, in heterotic string theory, by investigating the possible modifications in superstring effective theories of the $d = 10$, $N = 1$ Einstein-Yang-Mills supergravity theory, it has been shown that these plane geometries are exact solutions at all order of the string tension parameter [9]. Later, exact solutions describing colliding impulsive gravitational waves in dilaton gravity have been obtained [10] (these solutions have the Bell-Szekeres solution of general relativity [11] as a limiting case) and other ones have been derived by using the method of the harmonic mapping combined with the algebra associated with the group $SL(2, \mathbb{R})$ [12].

In this paper, we describe some plane geometric features appearing in a scalar tensor theory of gravity such as the Brans-Dicke theory and consider also the coupling of this theory to electromagnetism. More precisely, the purpose of this work is to study impulsive gravitational waves conjointly with null shells of matter and electromagnetic shock waves. In general relativity, a solution describing the collision of a scalar plane wave with either another scalar plane wave, or an impulsive gravitational wave or a neutrino wave has been obtained [13]. On the other hand, planar domain walls solutions coupled to the Brans-Dicke field have been given [14] [5]. Here, we describe situations in which an impulsive gravitational wave, a null shell of matter and possibly an electromagnetic shock wave coexist on the same null hypersurface. For that purpose, we use a formalism recently developped [5] which provide, in the context of scalar-tensor theories of gravity, a unified description of all types of hypersurfaces (timelike, spacelike and lightlike) and allows the space-time coordinates to be chosen freely and independently on both sides of the shell in agreement with their own symmetries.

The paper is organized as follows. In section 2, we review the shell formalism and summarize the principal results obtained in [5] for the null case in the Brans-Dicke theory and also some additional results obtained in [15] about the general separation between a null shell and a wave. In section 3, the Brans-Dicke vacuum case is considered. In the context of plane symmetry, an exact solution of the Brans-Dicke field equations is presented and is glued to the Minkowski space-time along a null hypersurface. This re-attachement is accomplished by making a suitable identification of the coordinates on both sides of the shell in terms of an arbitrary shift function. A null shell of matter and an impulsive gravitational wave are shown to coexist on this null hypersurface and the surface energy density of the shell and the wave amplitude are determined in the homogenous case. The inhomogenous case is also discussed. In section 4, the electrovacuum case is considered. An exact plane symmetric solution of the Brans-Dicke's field equations in the presence of a Maxwell field is given and it is found to present a true curvature singularity. This solution is glued to the vacuum solution of section 3 along a null hypersurface and represents the history of the wave fronts of a gravitational impulsive wave and of an electromagnetic shock wave propagating with and decoupled from a null shell of matter carrying no surface-current. Finally, it is remarked that the given exact solution can be interpreted as the interaction resulting from the collision of two electromagnetic shock waves of constant aligned polarization propagating in a conformally flat background and approaching from opposite directions. After the collision, two impulsive gravitational waves are produced and coexist with the original electromagnetic shock waves. This provides the analog in the Brans-Dicke theory of the Bell-Szekeres solution of general relativity [11] which is known to be diffeomorphic to the Bertotti-Robinson space-time [17] [16]. A difference arises however because the interaction region is not conformally flat as a consequence of the scattering of the gravitational waves and furthermore because of the presence of a true curvature singularity.

2 General formalism for a null shell in Brans-Dicke theory

In this section are summarized the main properties of a singular null hypersurface in the Brans-Dicke theory. A complete description of an arbitrary singular hypersurface (timelike, spacelike or lightlike) in the scalar-tensor theories of gravity, of which the Brans-Dicke theory is a particular case, having already presented in a recent paper [5], we only recall the results which will be useful for the present work and refer the reader to [5] for more details. Furthermore, more information on the justification of the separation between a shell part and a wave part which generally occurs on a lightlike hypersurface can be found in [15].

In the Brans-Dicke theory, it is well-known that two conformally related met-

rics can be used : the Jordan-Fierz metric usually referred to as the physical metric (because the matter fields are minimally coupled to it and the stress-energy tensor is conserved relatively to it), and the Einstein metric, where an Einstein-Hilbert term is recovered in the action (in the Einstein frame, the stress-energy tensor of the matter fields is not conserved). These frames provide two equivalent descriptions and as in [5], we shall work in the Einstein-frame where the equations of a shell take a simpler form. The space-time M in the Brans-Dicke theory is endowed with a pair $(g_{\mu\nu}, \varphi)$ where $g_{\mu\nu}$ is the space-time metric and φ a scalar field. In the presence of matter represented by a stress-energy tensor $T_{\mu\nu}$, the field equations for the pair $(g_{\mu\nu}, \varphi)$ are in the Einstein frame

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) + 2\,\partial_\mu\varphi\partial_\nu\varphi \tag{1}$$

$$\Box\varphi = -4\pi G\,\alpha T \tag{2}$$

where $\Box$ stands for the usual d'Alambertian operator associated with the metric $g_{\mu\nu}$ and where T represents the trace of $T_{\mu\nu}$ with respect to $g_{\mu\nu}$. In (2), α is the constant coupling factor of the scalar field φ to matter and it is related to the usual Brans-Dicke parameter ω by $\alpha^2 = (2\omega + 3)^{-1}$. In order to get the physical or Jordan-Fierz metric $\tilde{g}_{\mu\nu}$, one simply has to do the conformal transformation $\tilde{g}_{\mu\nu} = e^{\alpha\varphi} g_{\mu\nu}$.

Let us consider a null hypersurface Σ resulting from the isometric soldering of two isometric null hypersurfaces Σ^+ and Σ^- respectively embedded in the space-times M^+ and M^-, and further identify Σ as a null hypersurface in the manifold $M^+ \cup M^-$. Let us introduce two local systems of coordinates, $\{x_-^\mu\}$ in M^- and $\{x_+^\mu\}$ in M^+, the indices μ running from 0 to 3. Relatively to those systems of coordinates, we denote by $g_{\mu\nu}^+$ and $g_{\mu\nu}^-$ the components of the metric tensor , by φ^+ and φ^- the scalar field, and by $T_{\mu\nu}^+$ and $T_{\mu\nu}^-$ the components of the matter stress-energy tensors in each domain M^+ and M^-. The field equations (1)-(2) hold separetly in M^+ and M^- for the two sets $(g_{\mu\nu}^+, \varphi^+, T_{\mu\nu}^+)$ and $(g_{\mu\nu}^-, \varphi^-, T_{\mu\nu}^-)$ respectively. In the present work, Σ represents the history of a shell and/or an impulsive wave, and the metric and the scalar field are only C^0 on it i.e. $[g_{\mu\nu}] = [\varphi] = 0$ but $[\partial_\alpha g_{\mu\nu}] \neq 0$ and $[\partial_\alpha\varphi] \neq 0$ where the brackets denote the jump across Σ of the quantity contained therein.

Let us introduce a set of intrinsic coordinates ξ^a ($a = 1, 2, 3$) on Σ, the corresponding tangent basis vectors $e_{(a)} = \frac{\partial}{\partial\xi^a}$ and the common induced metric g_{ab} on Σ from M^+ and M^-. In the case of a lightlike hypersurface, the normal n is null and tangent to the surface. In order to describe how the hypersurface is embedded in spacetime, we need to introduce a transverse vector N on Σ such that

$$N.n = \eta^{-1} \qquad , \tag{3}$$

where η is a given non-vanishing smooth function on Σ. This transversal is uniquely defined by (3) up to a tangential displacement. In order to make sure

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that we consider the same normal and transversal on both sides, we impose the relations

$$\left[n.e_{(a)} \right] = \left[N.e_{(a)} \right] = [N.N] = 0 \qquad . \tag{4}$$

We define the transverse extrinsic curvature of Σ on the $+$ side and $-$ side by

$$\mathcal{K}_{ab}^{\pm} = - N.\nabla_{e_{(b)}} e_{(a)}|^{\pm} \qquad . \tag{5}$$

Because the metric is only continuous across Σ, the transverse extrinsic curvature is discontinuous across Σ and its jumps are denoted

$$\gamma_{ab} = 2 \left[\mathcal{K}_{ab} \right] \qquad . \tag{6}$$

It can be shown that the 3-tensor γ_{ab}'s is independent on the choice of the transverse vector field N and provides an intrinsic description of the embedding of Σ in the manifold $M^+ \cup M^-$. It can be extended (in a non unique way) into a 4-tensor $\gamma_{\mu\nu}$ in either M^+ or M^- by requiring its projection on Σ to be γ_{ab} i.e. $\gamma_{\mu\nu} e_{(a)}^{\mu} e_{(b)}^{\nu} = \gamma_{ab}$.

Introducing these results into the field equations (1)-(2), one can show that the matter stress-energy tensor $T^{\mu\nu}$ admits a singular Dirac part $S^{\mu\nu}$ concentrated on Σ and takes the following general form

$$T^{\mu\nu} = \chi \, S^{\mu\nu} \, \delta(\Phi) + T_{\pm}^{\mu\nu} \qquad , \tag{7}$$

where $\Phi(x^{\mu}) = 0$ is the equation of Σ in a local system of coordinates and χ is a normalization factor such that $n = \chi^{-1} \nabla\Phi$. The surface stress-energy tensor $S^{\mu\nu}$ is given by [5]

$$16\pi \, \eta^{-1} \, S^{\mu\nu} = 2 \, \gamma^{(\mu} n^{\nu)} - \gamma \, n^{\mu} n^{\nu} \qquad , \tag{8}$$

with

$$\gamma^{\mu} = \gamma^{\mu\nu} n_{\nu} \; ; \; \gamma = g^{\mu\nu} \gamma_{\mu\nu} \qquad , \tag{9}$$

These quantities can be calculated on either side of Σ and it can be proven that the expression (8) is independent of the extension from γ_{ab} to $\gamma_{\mu\nu}$. The surface-stress energy tensor $S^{\mu\nu}$ is purely tangential, $S^{\mu\nu} n_{\nu} = 0$, and it can also be expressed in the tangent basis $e_{(a)}$ as an intrinsic 3-tensor S^{ab} given by

$$16\pi G \, \eta^{-1} \, S^{ab} = \left(g_*^{ac} \, l^b l^d + l^a l^c \, g_*^{bd} - l^a l^b \, g_*^{cd} \right) \gamma_{cd} \tag{10}$$

where the l^a's are such that

$$n = l^a e_{(a)} \qquad , \tag{11}$$

and satisfy $g_{ab} l^b = 0$. The quantity g_*^{ab} is defined by

$$g_*^{ab} \, g_{bc}^* = \delta_c^a - \eta l^a N_c \qquad . \tag{12}$$

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The Weyl tensor of the global space-time $M^+ \cup M^-$ has in general a Dirac part concentrated on Σ which is given as in general relativity [18] by

$$C^{\alpha\beta}_{\mu\nu} = \{\, 2\eta\, n^{[\alpha}\gamma^{\beta]}_{[\mu}n_{\nu]} - 16\pi\, \delta^{[\alpha}_{[\mu}S^{\beta]}_{\nu]} + \frac{8}{3}\pi\, S^\lambda_\lambda \delta^{\alpha\beta}_{\mu\nu} \,\}\, \chi\, \delta(\Phi) \qquad . \tag{13}$$

An important result is that the γ_{ab}'s split into two independent parts, which are separetely associated with a shell and with a wave. It is easy to see from (8) or (10) that only the components $\gamma_{ab}l^b$ and $\gamma = g^{\mu\nu}\gamma_{\mu\nu} = g^{cd}_*\gamma_{cd}$ of the γ_{ab}'s contribute to the expression of the stress-energy tensor. This leaves two independent components denoted by $\hat{\gamma}_{ab}$ representing an impulsive gravitational wave and related to the two degrees of freedom of polarization of the wave. The expression of the $\hat{\gamma}_{ab}$'s is [5] [15]

$$\hat{\gamma}_{ab} = \gamma_{ab} - \frac{\gamma}{2}g_{ab} + \eta(N_a\gamma_{bc} + N_b\gamma_{ac})l^c \qquad , \tag{14}$$

The impulsive gravitational wave propagates with the shell in the direction n^μ in space-time and with Σ as history of its wave-fronts.

In the case when an electromagnetic field $F_{\mu\nu}$ is present, it may be discontinuous across Σ and the discontinuities of the field are related to the existence of an electromagnetic shock wave and to surface-currents (note that this is also valid for an arbitrary gauge field -see [5]). The electromagnetic potential A_μ is continuous across Σ *i.e.* $[A_\mu] = 0$, but its transverse derivatives are not and one introduces a vector λ^ν such that

$$[\partial_\mu A_\nu] = \eta n_\mu \lambda_\nu \qquad . \tag{15}$$

The jump of the electromagnetic field across Σ is therefore

$$[F_{\mu\nu}] = \eta(n_\mu\lambda_\nu - n_\nu\lambda_\mu) \qquad . \tag{16}$$

From the Maxwell equations, it can be shown that there exist a surface current j^μ such that

$$4\pi j^\mu = \eta(\lambda.n)n^\mu \qquad . \tag{17}$$

This expression shows that j^μ is purely tangential to Σ and that only the part $\lambda.n$ of λ_μ contributes to the surface current. The remaining part

$$\hat{\lambda}^\mu = \lambda^\mu - \eta(\lambda.n)\,N^\mu \qquad , \tag{18}$$

characterizes the electromagnetic shock wave.

3   Pure vacuum

In this section, we first solve the Brans-Dicke's field equations (1)-(2) in the case of vacuum ($T^{\mu\nu} = 0$) and plane symmetry. Then we apply our solution to the

description of a plane null shell and impulsive gravitational wave. It is convenient to take the metric in the form given by Szekeres [19]

$$ds^2 = -2e^{-M}dudv + e^{-U}(e^V dx^2 + e^{-V}dy^2) \quad , \tag{19}$$

where the functions M, U, V and the scalar field φ only depend on the null coordinates (u, v), and (x, y) are spacelike coordinates - we use the ordering u, v, x, y and greek indices range from 0 to 3. The form of the field equations (1)-(2) with the metric (19) and an arbitrary stress-energy tensor $T^{\mu\nu}$ is given is appendix A. The space-time defined by the metric (19) contains plane waves propagating along the null hypersurfaces $u = constant$ and $v = constant$ as demonstrated by Bondi, Pirani and Robinson [20][8].

In vacuum, the field equations reduce to the following set of equations

$$2\,U_{uu} - U_u{}^2 - V_u{}^2 + 2\,M_u\,U_u \;=\; 4\,\varphi_u{}^2 \tag{20}$$

$$2\,U_{vv} - U_v{}^2 - V_v{}^2 + 2\,M_v\,U_v \;=\; 4\,\varphi_v{}^2 \tag{21}$$

$$U_u\,U_v - U_{uv} \;=\; 0 \tag{22}$$

$$2\,V_{uv} - U_u\,V_v - U_v\,V_u \;=\; 0 \tag{23}$$

$$U_{uv} + 2\,M_{uv} - V_u\,V_v \;=\; 4\,\varphi_u\,\varphi_v \tag{24}$$

$$2\,\varphi_{uv} - U_u\,\varphi_v - U_v\,\varphi_u \;=\; 0 \tag{25}$$

where the indices u and v refer to partial derivatives with respect to these variables. Equation (22) can immediately be integrated as

$$e^{-U} \;=\; f(u) + g(v) \tag{26}$$

where f and g are arbitrary functions and, following Szekeres [19], the general solution of (23) is

$$V = \lambda_1 \tanh^{-1}\left[\sqrt{\frac{\frac{1}{2} - f}{\frac{1}{2} + g}}\right] + \lambda_2 \tanh^{-1}\left[\sqrt{\frac{\frac{1}{2} - g}{\frac{1}{2} + f}}\right] \tag{27}$$

where λ_1 and λ_2 are two arbitrary constants. Moreover, as the equation (25) for the scalar field φ is similar to (23), one gets for the scalar field [13]

$$\varphi = \lambda_3 \tanh^{-1}\left[\sqrt{\frac{\frac{1}{2} - f}{\frac{1}{2} + g}}\right] + \lambda_4 \tanh^{-1}\left[\sqrt{\frac{\frac{1}{2} - g}{\frac{1}{2} + f}}\right] + \varphi_0 \tag{28}$$

where λ_3, λ_4 and φ_0 are three arbitrary constants. In order for (26-28) to represent well-defined solutions, the functions f and g must satisfy $0 < f < \frac{1}{2}$ and $0 < g < \frac{1}{2}$. Equation (20) then gives

$$M_u = -\frac{f''(u)}{f'(u)} - \frac{f'(u)}{2(f+g)}\left[\alpha^2\left(\frac{\frac{1}{2}+g}{\frac{1}{2}-f}\right) + \beta^2\left(\frac{\frac{1}{2}-g}{\frac{1}{2}+f}\right) + \gamma\frac{\sqrt{\frac{1}{2}+g}\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}-f}} - 1\right] \tag{29}$$

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where we have introduced the scalars

$$\alpha^2 = \frac{\lambda_1^2}{4} + \lambda_3^2 \ , \ \ \beta^2 = \frac{\lambda_2^2}{4} + \lambda_4^2 \ , \ \ \gamma = \frac{\lambda_1 \lambda_2}{4} + \lambda_3 \lambda_4 \tag{30}$$

A similar expression can be obtained for M_v from equation (21) by interchanging $f(u)$ with $g(v)$ and α with β. Then, integrating M_u and M_v, one gets

$$
\begin{aligned}
M \ = \ & -\log(kf'g') - \frac{\alpha^2 + \beta^2 + 2\gamma - 1}{2} \log(f+g) + \frac{\alpha^2}{2} \log\left((\frac{1}{2} - f)(\frac{1}{2} + g) \right) \\
& + \frac{\beta^2}{2} \log\left((\frac{1}{2} + f)(\frac{1}{2} - g) \right) + 2\gamma \log D
\end{aligned}
$$

where $D = \sqrt{\frac{1}{2} - f}\sqrt{\frac{1}{2} - g} + \sqrt{\frac{1}{2} + f}\sqrt{\frac{1}{2} + g}$ and k is an arbitrary constant. This expression is similar to general relativity -see [8]- but the constants α, β and γ now include those of the Brans-Dicke scalar field.

It can be checked that the remaining equation (24) is identically satisfied when U, V, M and φ are replaced by the above expressions. Hence, these expressions constitute a plane symmetric solution of the vacuum Brans-Dicke field equations. As we shall later be concerned with a plane wave or a null shell along $u = const.$, we now consider the case where the metric and the scalar field only depend on u. The corresponding solution is obtained by putting $g = \frac{1}{2}$ in (27) and (28) and taking $\alpha^2 = \frac{\lambda_1^2}{4} + \lambda_3^2$, and $\beta = \gamma = 0$. One then gets

$$e^{-U} \ = \ f + \frac{1}{2} \tag{31}$$

$$e^V \ = \ \left(\frac{1 + \sqrt{\frac{1}{2} - f}}{1 - \sqrt{\frac{1}{2} - f}} \right)^{\frac{\lambda_1}{2}} \tag{32}$$

$$e^{-M} \ = \ kf'(u)\frac{\left(\frac{1}{2} + f\right)^{\frac{1}{2}\left(\alpha^2 - 1\right)}}{\left(\frac{1}{2} - f\right)^{\alpha^2}} \tag{33}$$

$$e^{\varphi} \ = \ e^{\varphi_0}\left(\frac{1 + \sqrt{\frac{1}{2} - f}}{1 - \sqrt{\frac{1}{2} - f}} \right)^{\frac{\lambda_3}{2}} \ . \tag{34}$$

Let us now consider two space-times M^+ and M^- separated by a null hyper-surface Σ located at $u = 0$. We suppose that M^+ has coordinates (u, v^+, x^+, y^+) and a metric and a scalar field given by (31-34). M^- is the Minkowski space-time with coordinates (u, v, x, y), a metric given by (19) with $M = U = V = 0$ and a constant scalar field (we drop the minus indices on any quantity belonging to M^-). A Brans-Dicke solution can be re-attached to the Minkowski space-time only if the scalar field takes this same constant value on the surface of junction.

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The two space-times are glued along the null hypersurface Σ, $u = 0$, by making the identification

$$(0, v^+, x^+, y^+) = (0, v - F(x, y), x, y) \tag{35}$$

where F is an arbitrary function of the coordinates x and y producing a shift in the null coordinate v tangent to the hypersurface. This identification was previously introduced by Penrose to generate a plane impulsive gravitational wave in the Minkowski space-time [21].

The condition of continuity of the metric across Σ requires that

$$f(0) = \frac{1}{2} \tag{36}$$

In order for M expressed in (33) to be defined at $u = 0$, we choose the arbitrary function $f(u)$ such that

$$f(u) = \frac{1}{2} - (\mu u)^\rho, \qquad \rho > 0 \tag{37}$$

with

$$\rho \left(1 - \frac{\alpha^2}{2} \right) = 1 \ , \ 1 \leq \alpha^2 < 2 \ . \tag{38}$$

Although no condition of continuity need to be imposed on M as it does not contribute to the induced metric on Σ, one chooses the arbitrary constant $k = -(\mu\rho)^{-1}$ and obtains for M

$$e^{-M} = (1 - (\mu u)^\rho)^{\frac{\alpha^2 - 1}{2}} \tag{39}$$

making M equal to its Minkowski value $M = 0$ at $u = 0$. Finally, using (34), the continuity of the scalar field across Σ is automatically satisfied provided that the constant value of the scalar field in M^- is φ_0.

We take $\xi^a = (v, x, y)$ with $a = 1, 2, 3$, as intrinsic parameters on Σ, and choose the normal to be $n = e_{(1)}$. We obtain from (11) $l^a = \delta_1^a$. The induced metric is $g_{ab} = diag(0, 1, 1)$ and one may take for its 'inverse' (12), $g_*^{ab} = diag(0, 1, 1)$. A convenient choice for the transversal N corresponds to $N.n = -1$, $N.e_{(2)} = N.e_{(3)} = 0$, $N.N = 0$, thus leading to components equal to $N^\alpha = (1, 0, 0, 0)$ in M^-, and

$$N_+^\alpha = \left(1, \frac{F_x^2 + F_y^2}{2}, -F_x, -F_y \right) \quad , \tag{40}$$

in M^+. Introducing these results into (5), one obtains the transverse extrinsic curvature $\mathcal{K}_{ab}^\pm$ on each side of the shell. As $\mathcal{K}_{ab}^-$ vanishes, one gets from (6), $\gamma_{ab} = 2\mathcal{K}_{ab}^+$ and the non-zero components of γ_{ab} are equal to

$$\begin{aligned} \gamma_{22} &= -2F_{xx} \\ \gamma_{33} &= -2F_{yy} \\ \gamma_{23} &= \gamma_{32} = -2F_{xy} \end{aligned} \tag{41}$$

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Therefore the surface stress-energy tensor (10) is of the form, $-S^{ab} = \sigma l^a l^b$, and has only one non-vanishing component equal to

$$16\pi G S^{11} = \gamma_{22} + \gamma_{33} = -2\,\Delta F \tag{42}$$

where Δ is the Laplacian in the 2-dimensional flat space (x, y). These results show that the shell has no shear and is only characterized by a surface energy density σ equal to

$$\sigma = \frac{\Delta F}{8\pi G} \qquad . \tag{43}$$

which requires $\Delta F \geq 0$ to ensure the positivity of the energy density.

On the other hand the wave part (14) of γ_{ab} has the only non-vanishing components

$$\hat{\gamma}_{22} = -\hat{\gamma}_{33} = -F_{xx} + F_{yy} \qquad , \tag{44}$$

$$\hat{\gamma}_{23} = -2F_{xy} \qquad . \tag{45}$$

The above results show that a null shell and an impulsive gravitational wave generally coexist and that their properties are characterized by the shift function $F(x, y)$. The shell and the wave propagate independently and have the same null hypersurface Σ as world sheet. The homogeneous case is obtained by taking

$$F(x, y) = \frac{A}{4}(x^2 + y^2) - \frac{B}{4}(x^2 - y^2) - \frac{C}{2}xy \tag{46}$$

where A, B and C are three constants with $A \geq 0$. The shell has a constant energy density such that $8\pi G\sigma = A$ and for the wave we have $\hat{\gamma}_{22} = \hat{\gamma}_{33} = B$, $\hat{\gamma}_{23} = C$. If $A = 0$, we have only a wave and only a shell if $B = C = 0$. Inhomogenous plane shells and waves are obtained if the shift function $F(x, y)$ is arbitrary. The expansion ρ and shear σ for the null geodesics with tangent vector N transverse to the hypersurface Σ vanish on the M^- side, and on the M^+ side, they are equal to

$$\rho^+ \;=\; N_{\mu;\nu} m^\mu \bar{m}^\nu \;=\; -\frac{1}{2}(F_{xx} + F_{yy}) = -4\pi G\sigma \tag{47}$$

$$\sigma^+ \;=\; N_{\mu;\nu} m^\mu m^\nu \;=\; -\frac{1}{2}(F_{xx} - F_{yy}) - iF_{xy} = \frac{1}{2}(\hat{\gamma}_{22} + i\hat{\gamma}_{23}) \tag{48}$$

where we have used the null tetrad $(n, N, m, \bar{m})$ with $m = (e_{(2)} + ie_{(3)})/\sqrt{2}$ and where ; stands for the covariant derivative. Since $\rho^+ \leq 0$, the null congruence is focussed after having crossed the shell with energy density $\sigma \geq 0$ and from (48), one sees that the wave part is responsible for the shear of the null geodesics.

In this paper, we have glued the two space-times $M^\pm$ by making the identification (35) and have found that the shell is only characterized by its surface-energy density. When the relation between the coordinates v^+, x^+, y^+ and v, x, y takes a more general form, we expect the shell to admit also a surface pressure and anisotropic surface stresses.

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4 Electrovacuum

In this section, we first present an exact plane wave solution of the Brans-Dicke field equations in the presence of an electromagnetic field; this solution admits a curvature singularity. Then, using this solution, one builts an example of a plane impulsive gravitational wave propagating together with a null shell of matter and an electromagnetic shock wave on the same null hypersurface. Finally, one shows that our exact solution describes the interaction region resulting from the collision of two electromagnetic shock waves propagating in a Brans-Dicke conformally flat background. This provides the analogue of the Bell-Szekeres solution in general relativity [11] but the properties obtained here in the interaction region are completly different because of a presence of a curvature singularity.

As the electromagnetic stress-energy tensor $T_{\mu\nu}$ is traceless, the Brans-Dicke's field equations reduce to

$$R_{\mu\nu} = 8\pi\, G\, T_{\mu\nu} + 2\,\partial_\mu\varphi\,\partial_\nu\varphi \tag{49}$$

$$\Box\varphi = 0 \tag{50}$$

and they have to be solved simultanously with the Maxwell vacuum equations

$$\nabla_\mu F^{\mu\nu} = 0 \tag{51}$$

We still use the Szekeres form (19) of the metric as we still have the plane symmetry and we take for the electromagnetic potential the form recently introduced in [22] to describe plane gravitational waves in Bertotti-Robinson spacetimes [17] [16] [23]

$$A = \cos\theta\,\sin(au - bv)\,dx + \sin\theta\,\sin(au + bv)\,dy \qquad . \tag{52}$$

where a, b and θ are three arbitrary constants. Using the expressions (A.7-A.10) for the components of $T_{\mu\nu}$ in terms of the metric functions and the field equations (A.1-A.6), we obtain the following solution

$$U = M = -\log\left[\cos(au - bv)\,\cos(au + bv)\right] \qquad , \tag{53}$$

$$V = \log\left[\frac{\cos(au - bv)}{\cos(au + bv)}\right] \qquad , \tag{54}$$

$$\varphi = \frac{1}{\sqrt{2}}\,\log\left[\cos(au - bv)\,\cos(au + bv)\right] + \varphi_0\,,\ \varphi_0 = const \qquad . \tag{55}$$

The metric is then

$$ds^2 = \cos^2(au - bv)\,dx^2 + \cos^2(au + bv)\,dy^2 - 2\,\cos(au - bv)\,\cos(au + bv)\,dudv \tag{56}$$

and it can be checked that the electromagnetic potential (52) automatically satisfies the Maxwell's equations (51). The solution is well-defined for $-\pi/2a < u <$

$\pi/2a$ and $-\pi/2b < v < \pi/2b$ and there is a curvature singulatity on the hyper-surfaces $au \pm bv = \pi/2$ as it can be seen by calculating the curvature invariants R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$. For example, R is given by

$$R = \frac{2ab\sin(2au)\sin(2bv)}{\cos^{3/2}(au - bv)\cos^{3/2}(au + bv)} \tag{57}$$

and diverges at $au \pm bv = \pi/2$ including the 2-dimensional boundaries $\{u = \pi/2a, v = 0\}$ and $\{u = 0, v = \pi/2b\}$.

Introducing the following Newmann-Penrose null tetrad $(l, n, m, \bar{m})$ relatively to the space-time metric (56)

$$\begin{align}
l_\mu &= (0, \cos(au - bv)\cos(au + bv), 0, 0) \tag{58}\\
n_\mu &= (1, 0, 0, 0) \tag{59}\\
m_\mu &= \left(0, 0, \frac{1}{\sqrt{2}}\cos(au - bv), \frac{i}{\sqrt{2}}\cos(au + bv)\right) \tag{60}
\end{align}$$

one finds the following components of the electromagnetic field associated with the potential (52)

$$F_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & ae^{i\theta} & ae^{-i\theta} \\ 0 & 0 & -be^{i\theta}f^{-1} & -be^{-i\theta}f^{-1} \\ -ae^{i\theta} & be^{i\theta}f^{-1} & 0 & 0 \\ -ae^{-i\theta} & be^{-i\theta}f^{-1} & 0 & 0 \end{pmatrix} \tag{61}$$

where $f \equiv \cos(au - bv)\cos(au + bv)$. The electromagnetic invariant and pseudo invariant are respectively given by

$$\begin{align}
F_{\mu\nu}F^{\mu\nu} &= \frac{4ab\sin(2\theta)}{\cos(au - bv)\cos(au + bv)} \\
F_{\mu\nu}{}^*F^{\mu\nu} &= \frac{4ab\cos(2\theta)}{\cos(au - bv)\cos(au + bv)} \tag{62}
\end{align}$$

showing that the field is null when a or b are zero and that the electric and magnetic parts are parallel when $a = \pm b$. Moreover one sees that the electromagnetic field is like the curvature singular ot $au \pm bv = \pi/2$.

The null hypersurfaces $u = const.$ are generated by null geodesics with tangent $\partial/\partial v$. Their null generators have expansion ρ and shear σ given by

$$\rho = \frac{b\sin(2bv)}{2\cos(au - bv)\cos(au + bv)}, \quad \sigma = \frac{b\sin(2au)}{2\cos(au - bv)\cos(au + bv)} \tag{63}$$

One finds that there is an infinite focussing of the null congruence at the hyper-surfaces $au \pm bv = \pi/2$. These null geodesics are shear-free and expansion-free when $b = 0$ and in this case correspond to plane waves. The existence of these

plane waves is a similar result as first obtained in general relativity in ref. [22] using Bertotti-Robinson space-times where the same formulas (63) hold. Furthermore, when $b = 0$, the metric (56) is conformally flat. Similar properties hold for the null hypersurfaces $v = constant$ generated by the vector field $\partial/\partial u$.

The properties of the space-time described by the metric (56) are thus much different from the Bertotti-Robinson solution for the following reasons. The Bertotti- Robinson solution of general relativity is the only solution of the Einstein-Maxwell theory which is homogenous and has a homogenous non-null electromagnetic field [23] and moreover it is conformally flat and without curvature singularity. Here, the electromagnetic field is also non-null but non homogenous. Moreover, there is a curvature singularity and the space-time is not conformally flat when both a and b are non-zero.

Let us consider as in section 2, a null hypersurface Σ along which are glued two spacetimes M^+ and M^-, M^- being to the past of Σ. We take for M^- the vacuum solution of section 2 and for M^+ the above solution. The coordinates are (u, v_+, x_+, y_+) in M^+ with $u > 0$ and (u, v, x, y) in M^- with $u < 0$ -here again, we drop the minus indices for any quantity refering to M^-. The null hypersurface Σ is located at $u = 0$ represents the propagation of a null shell and/or an impulsive gravitational wave with an electromagnetic shock wave. The space-time metrics, the scalar fields and the electromagnetic potentials only depend on the u-coordinate and are given respectively by (31-32-34-39) with (37) in M^- and (52)-(56) with $b = 0$ in M^+ *i.e.*

$$ds_+^2 = \cos^2(au)\left(-2\,dudv^+ + dx_+^2 + dy_+^2\right) \tag{64}$$

$$\varphi_+ = \sqrt{2}\,\log\left[\cos(au)\right] + \varphi_0^+ \tag{65}$$

$$A_+ = \sin(au)(\cos\theta dx_+ + \sin\theta dy_+) \qquad . \tag{66}$$

As in section 2, the soldering of the two spacetimes along Σ is realized by making the identification

$$(0, v_+, x_+, y_+) = (0, v - F(x, y), x, y) \qquad , \tag{67}$$

and we take $\xi^a = (v, x, y)$ with $a = 1, 2, 3$, as intrinsic parameters on Σ, $e_{(a)} = \partial/\partial\xi^a$ being the corresponding tangent basis vectors and $n = e_{(1)}$ the normal to Σ. The induced metrics on Σ from M^+ and M^- are identical and equal to $g_{ab} = diag(0, 1, 1)$ and one takes for the "inverse" (12) $g_*^{ab} = diag(0, 1, 1)$. The electromagnetic potential vanishes on Σ. The continuity of the scalar field on Σ implies $\varphi_0^+ = \varphi_0$ and the continuity of the metric requires that $U^+(0) = V^+(0) = 0$ conditions which are trivially satisfied with our choice. A convenient choice for the transversal N corresponds to $N.n = -1$ and $N.e_{(2)} = N.e_{(3)} = N.N = 0$. Although the space-times $M^\pm$ are different from those of section 2, one gets the same components (40) and the same is true for the extrinsic curvature $\mathcal{K}_{ab}$: it vanishes in M^- and its jumps γ_{ab} is still given by (41). It follows that

the properties of the null shell and the gravitational wave are unchanged and characterized by the shift function $F(x, y)$. For the jump in the electromagnetic field across Σ, we get from (15)

$$\lambda_\nu = (0, 0, a \cos \theta, a \sin \theta) \qquad . \tag{68}$$

As $\lambda.n = 0$, the null hypersurface Σ does not carry any surface current. Therefore, it is simply the history of the wave-fronts of a plane electromagnetic shock wave which is entirely characterized by the vector λ_ν and of an plane impulsive gravitational wave and at the same time the history of a null shell of matter. In the context of general relativity, it was first pointed out in [22] (using a system of coordinates continuous across the shell) that such a coexistence was possible in Bertotti-Robinson space-times.

Finally, the exact solution (52-56) with $a > 0$ and $b > 0$ can be viewed as resulting from collision of two electromagnetic shock waves approching from opposite directions and with aligned polarization. Let us divide the space-time into four regions according to fig. 1.

In region I, the space-time is flat i.e. the metric is of the form (19) with $U = V = M = 0$ and the scalar field and the electromagnetic potential are zero. In region II, the metric, the scalar field and the electromagnetic field are given by

$$ds^2 = \cos^2(au) \left(-2 \, dudv + dx^2 + \, dy^2 \right) \tag{69}$$

$$\varphi = \sqrt{2} \, \log \left[\cos(au) \right] \tag{70}$$

$$A = \sin(au)(\cos \theta_1 dx + \sin \theta_1 dy) \qquad . \tag{71}$$

and in region III

$$ds^2 = \cos^2(bv) \left(-2 \, dudv + dx^2 + \, dy^2 \right) \tag{72}$$

$$\varphi = \sqrt{2} \, \log \left[\cos(bv) \right] \tag{73}$$

$$A = \sin(bv)(\cos \theta_2 dx + \sin \theta_2 dy) \qquad . \tag{74}$$

where θ_1 and θ_2 are arbitrary constants. The space-time regions II and III exhibit topological singularities at $v = \pi/2b$ and $u = \pi/2a$ respectively. These singularities are not curvature singularities since the curvature tensor remains finite on these hypersurfaces[1]. Moreover, all the curvature invariants vanish on these null boundaries.

Between the two regions I and II, the metric and the scalar field are C^1 and the electromagnetic potential C^0 across $\Sigma_1 = \{u = 0\}$. Therefore, there is no impulsive gravitational wave and no shell but an electromagnetic shock wave propagating along Σ_1. In a similar way, between regions I and III, the metric and

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[1]Actually, all the Riemann components vanish except $R_{uxux} = R_{uyuy}$ which have constant values $2a^2$ and $2b^2$ on each boundary respectively.

IV (u > 0, v > 0)

$\Psi_0 \neq 0$ $\Psi_2 \neq 0$ $\Psi_4 \neq 0$

$\varphi \, (u,v) \neq 0$

v

e.m. shock wave $\Phi_0 \sim \Theta(u)$
grav. imp. wave $\Psi_0 \sim \delta(u)$

e.m. shock wave $\Phi_2 \sim \Theta(v)$
grav. imp. wave $\Psi_4 \sim \delta(v)$

u

II (u < 0, v > 0)

All the $\Psi_i = 0$

$\Phi_2 \neq 0$

$\varphi \, (v) \neq 0$

III (u > 0, v < 0)

All the $\Psi_i = 0$

$\Phi_0 \neq 0$

$\varphi \, (u) \neq 0$

e.m. shock wave $\Phi_2 \sim \Theta(v)$

FLAT

I (u < 0, v < 0)

All the $\Psi_i = 0$

All the $\Phi_i = 0$

$\varphi = 0$

Σ_1

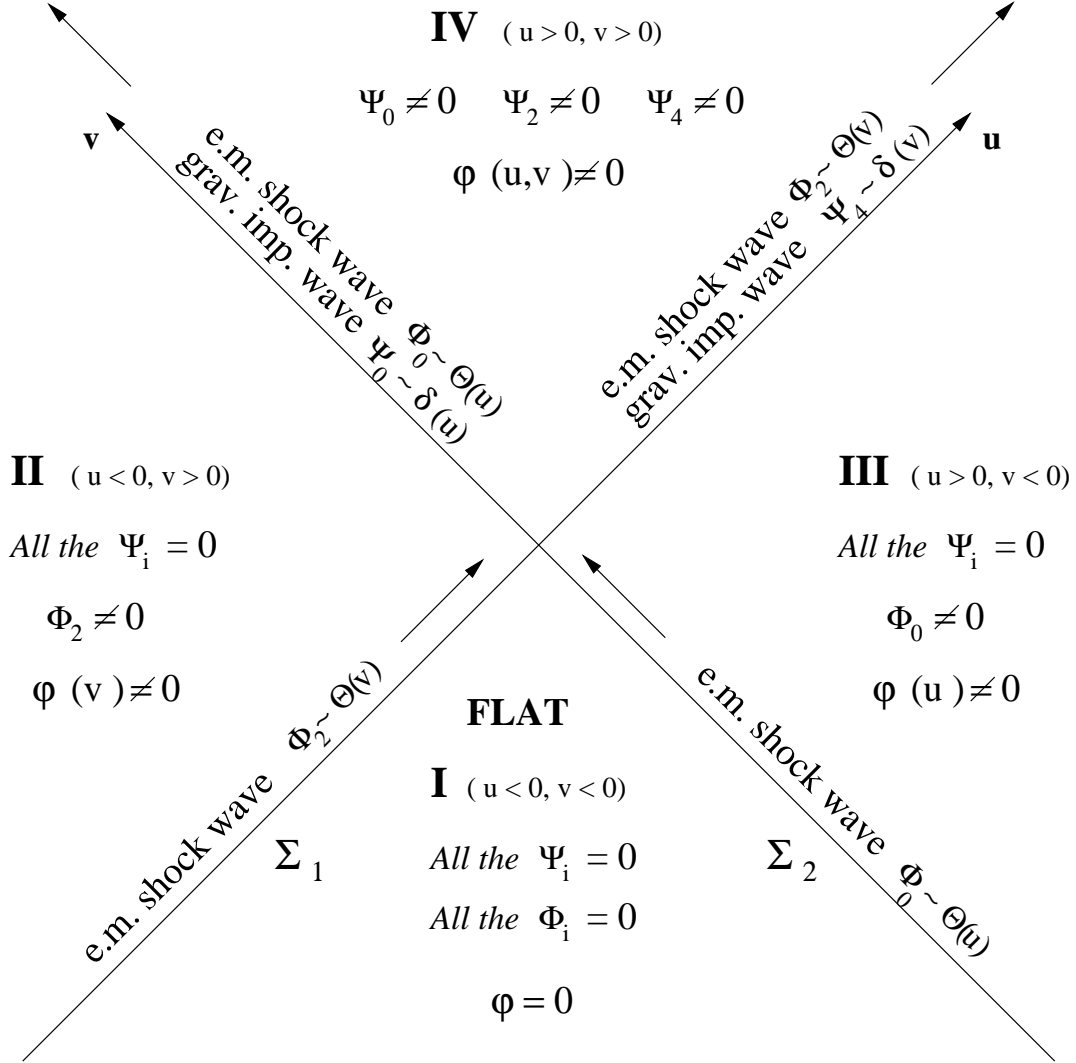Σ_2

e.m. shock wave $\Phi_0 \sim \Theta(u)$

Figure 1: collision of two plane electromagnetic (e.m.) shock waves giving two gravitational impulsive (grav.imp.) waves and two plane electromagnetic shock waves.

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the scalar field are C^1 across the boundary $\Sigma_2 = \{v = 0\}$ and the electromagnetic potential is only C^0 and there is only an electromagnetic shock wave propagating along Σ_2.

The space-time region IV-see fig.1- has the metric (56), the electromagnetic potential (52) and the scalar field (55). One now shows that this choice in region IV satisfies the boundary conditions for the re-attachement of the domains IV to II and IV to III along the null boundaries Σ_1 and Σ_2 respectively. The metric, the scalar field and the electromagnetic potential are continuous across Σ_1 between regions II and IV and across Σ_2 between regions III and IV. But there is a jump in the transverse derivatives of the metric across Σ_1 which proves the existence of an impulsive gravitational wave which has Σ_1 as history of its wave-fronts and propagating with an plane electromagnetic shock wave. The same arguments applied between regions III and IV lead to the conclusion that an impulsive gravitational wave and an electromagnetic shock wave propagate along Σ_2 with this null hypersurface as their history.

The interaction of two electromagnetic shock waves then produces two gravitational impulsive waves propagating with electromagnatic shock waves. This is the analog of te Bell-Szekeres solution in general relativity but the interaction region now exhibits a curvature singularity on the hypersurface $au \pm bv = \pi/2$ which extends back into the past to the topological singularities $v = \pi/2b$ of the region II and $u = \pi/2a$ in region III and thus act as "fold singularities" [8]. In region IV, the non-vanishing Newmann-Penrose components of the electromagnetic field on the null tetrad (58-60) are

$$\phi_0 = -\frac{a}{\sqrt{2}} g \ , \ \ \phi_2 = -\frac{b}{\sqrt{2}\, f^2} g \tag{75}$$

where $g \equiv \frac{e^{i\theta}}{cos(au-bv)} - i \frac{e^{i\theta}}{cos(au+bv)}$. A computation of the Weyl scalars of the region IV reveals that this region is not conformally flat. On the null tetrad (58-60), the non-vanishing Newmann-Penrose Weyl scalars are

$$\Psi_0 = b\tan(au)\Theta(u)\delta(v) + \frac{b^2}{2}(\tan^2(au+bv) - \tan^2(au-bv))\Theta(v) \tag{76}$$

$$\Psi_2 = -\frac{ab}{6}\frac{(\tan^2(au+bv) - \tan^2(au-bv))}{\cos(au-bv)\cos(au+bv)}\Theta(u)\Theta(v) \tag{77}$$

$$\Psi_4 = a\tan(bv)\Theta(v)\delta(u) + \frac{a^2}{2}(\tan^2(au+bv) - \tan^2(au-bv))\Theta(u) \tag{78}$$

where Θ stands for the Heaviside step function. One concludes that the region IV exhibits gravitational shock waves which are absent in the Bell-Szekeres solution where the four regions are all conformally flat. Moreover, there exists a "Coulomb" component Ψ_2 in the interaction region which can be interpreted as a scattering effect of the two gravitational shock waves.

Acknowledgments

References

[1] Kaluza T. 1926 *Sitz. Preuss. Akad. Wiss.* **1921** 966
Klein O. 1926 *Z. Phys.* **37** 895
Cho Y. 1992 *Phys. Rev. Lett.* **68** 3133

[2] M. Fierz 1956 *Helv.Phys.Acta* **29** 128
P. Jordan 1959 *Z.Phys.* **157** 112
C. Brans and R.H. Dicke 1961 *Phys.Rev.* **124** 925

[3] T. Damour and G. Esposito-Farese 1993 *Class.Quantum Grav* **9** 6442
Berkin A. L. and Hellings R. W. 1994 *Phys.Rev.D* **49** 6442
Damour T. and Esposito-Farese G. 1996 *Phys.Rev.D* **54** 1474

[4] Green M. G., Schwartz J. H., and Witten E. 1987 *Superstring Theory* (Cambridge: Cambridge University press)

[5] Barrabès C. and Bressange G.F. 1997 *Class. Quantum Grav.* **14** 805

[6] La D. and Steinhardt P. 1989 *Phys. Rev. Lett* **62** 376

[7] Ehlers J. and Kundt W. 1962 *Gravitation: an introduction to current research* ed L. Witten (New York: Wiley)

[8] Griffiths J.B. 1991 *Colliding Plane Waves in General Relativity* (Oxford: Claradon Press)

[9] Güven R. 1987 *Phys. Lett.* B **191** 275

[10] Gürses M. and Sermutlu E. 1995 *Phys.Rev.* D **52** 2

[11] Bell P. and Szekeres P. 1974 *Gen. Rel. Grav.* **5** 275-86

[12] Bretón N Matos T. and García A. 1996 *Phys. Rev.* D **53** 1868

[13] Wu Z. C. 1982 *J. Phys.* A **15** 2429

[14] Schmidt H-J. and Wang A. 1993 *Phys.Rev.* D **47** 4425

[15] Barrabès C., Bressange G.F. and Hogan P.A. 1997 *Phys. Rev.* D **55** 3477

[16] T. Levi-Civitta 1917 *R. C. Acad. Lincei* **26** 519

[17] Bertotti B. 1959 *Phys. Rev.* **116** 1331-3
Robinson I. 1959 *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys.* **7** 351-2

[18] Barrabès C. and Israel W. 1991 *Phys. Rev.* D **43** 1129

[19] Szekeres P. 1972 *J. Math. Phys.* **13**, 286-94

[20] Bondi H. Pirani F.A.E. and Robinson I. 1959 *Proc. Roy. Soc.* A **251** 519-33

[21] Penrose R. 1972 in *General Relativity: Papers in Honour of J. L. Synge* ed O'Raifeartaigh (Oxford: Clarandon Press)

[22] Hogan P.A. 1997 *Plane waves and Bertotti-Robinson space-times*, University College Dublin, preprint.

[23] Kramer D. Stephani H. Herlt E. and MacCallum M. 1980 *Exact solutions of Einstein's field equations*, ed Schmutzer E. (Cambridge: Cambridge university press)

Appendix A

When the metric $g_{\mu\nu}$ has the Szekeres form (19) and the scalar field φ only depends on the null coordinates u, v, the field equations (1)-(2) reduce to

$$R_{uu} = U_{uu} + U_u M_u - \frac{1}{2}(U_u^2 + V_u^2) = 8\pi\, T_{00} + 2\,(\partial_u\varphi)^2 \qquad (A.1)$$

$$R_{vv} = U_{vv} + U_v M_v - \frac{1}{2}(U_v^2 + V_v^2) = 8\pi\, T_{11} + 2\,(\partial_v\varphi)^2 \qquad (A.2)$$

$$R_{uv} = M_{uv} + U_{uv} - \frac{1}{2}(U_v U_u + V_u V_v) = 2\,\partial_u\varphi\,\partial_v\varphi \qquad (A.3)$$

$$R_{xx} = e^{M-U+V}\left\{V_{uv} + U_u U_v - U_{uv} - \frac{1}{2}(U_u V_v + U_v V_u)\right\} = 8\pi\, T_{22} \quad (A.4)$$

$$R_{yy} = e^{M-U-V}\left\{-V_{uv} + U_u U_v - U_{uv} + \frac{1}{2}(U_u V_v + U_v V_u)\right\} = 8\pi\, T_{33} \quad (A.5)$$

and

$$\Box\varphi = e^M\left(-2\,\partial_{uv}\varphi + U_v\,\partial_u\varphi + U_v\,\partial_u\varphi\right) = -4\pi\alpha T \qquad (A.6)$$

where $T = g^{\mu\nu}T_{\mu\nu}$. In this space-time, the stress-energy tensor $T_{\mu\nu}$ of the electromagnetic field corresponding to the potential (52) of the form $A = \lambda dx + \mu dy$, is diagonal and its components are given by

$$4\pi\, T_{00} = e^{U-V}(\lambda_u{}^2 + e^{2V}\mu_u{}^2) \qquad (A.7)$$

$$4\pi\, T_{11} \;=\; e^{U+V}(\lambda_v{}^2 + e^{2V}\mu_v{}^2) \tag{A.8}$$

$$4\pi\, T_{22} \;=\; -e^{M}\left(\lambda_u\lambda_v - e^{2V}\mu_u\mu_v\right) \tag{A.9}$$

$$4\pi\, T_{33} \;=\; -e^{M}\left(\mu_u\mu_v - e^{-2V}\lambda_u\lambda_v\right) \tag{A.10}$$